# TURBULENT FLOW OVER A BACKWARD FACING STEP USING PENALTY AND EQUAL-ORDER METHODS

Roald Akberov[1], Darrell W. Pepper[2], and Yitung Chen[1]
[1]Nevada Center for Advanced Computational Methods, University of Nevada Las Vegas
[2]Interim Dean, College of Engineering, University of Nevada Las Vegas

## ABSTRACT

Two-dimensional steady incompressible turbulent flow over a backward-facing step was calculated using equal-order and penalty function finite element methods. The standard k-ε turbulence model with wall functions was applied. Results from both of the methods for pressure-velocities coupling look pretty similarly to each other. The greatest discrepancy was observed for the velocity vector plots behind the step. While the equal order method showed physically correct behavior of velocity near the walls, the penalty function method gave oscillations for the velocity component parallel to the wall.

## NOMENCLATURE

E roughness parameter
**f** body force
H step height
k turbulent kinetic energy (TKE)
**n** outward unit normal vector on the boundary
p pressure
P(**u**) TKE production term
Re Reynolds number
t time
$\hat{\mathbf{t}}$ unit tangent vector on the boundary
**u** velocity vector
$u_\tau$ friction velocity
$v, w$ test functions
ε TKE dissipation rate
κ von Karman constant
λ penalty parameter
ν kinematic viscosity
$\tau_w$ wall shear stress
$\nabla$ gradient operator
$\nabla \cdot$ divergence operator

Subscripts

T turbulent
w values on the boundary
∞ free-stream value
0 initial condition

Superscripts

\+ dimensionless (wall functions)
n previous time step
n+1 current time step
* intermediate time step

## INTRODUCTION

It was found out back in early 1970's (Hood and Taylor, 1974) that when the Galerkin method is used together with the finite element method for solving incompressible fluid flow problems which are described by continuity and Navier-Stokes equations, mixed interpolation must be applied for approximating velocities and pressure within each element with interpolating basis functions for pressure of at least one order lower than for velocities. The equal-order interpolation leads to non-physical oscillations in all parameters of the flow and disstability of a numerical solution. The penalty function method represents one of the methods with mixed interpolation in which the right-hand side of the continuity equation which is normally equal to zero is replaced by pressure over a very large penalty parameter. This gives an explicit relation between velocities and pressure. Then the integral corresponding to the pressure gradient term in the Navier-Stokes equations can be integrated by parts and pressure from the modified continuity equation can be substituted to the resulting integral. Therefore, pressure is eliminated from the Navier-Stokes equations and this allows to develop a stable algorithm which would require solving only for velocities and pressure can be calculated using the modified continuity equation. Since mixed interpolation functions must be applied for approximating velocities and pressure, reduced numerical integration of the integrals corresponding to penalty terms in the Navier-Stokes equations must be used (Zienkiewicz and Taylor, 1989).

The other group of methods for pressure-velocity coupling is a so-called group of equal-order methods which implies time marching procedure where calculation of pressure is fulfilled through some Poisson-like equations. The most cited method of this group is the SIMPLE method by Patankar and Spalding (1972) in which a Poisson equation for pressure correction is introduced. After pressure corrections are found, velocities and pressure on a current time step can be updated. The revised version of this method, called

SIMPLER, (Patankar, 1980) updates pressure from a Poisson equation but not an algebraic one and can be included into the more common class of methods for pressure-velocity coupling called projection methods. The first projection method was proposed by Chorin (1968) and some further modifications into the method were made by Gresho et al (1984). The method was successfully applied for solving laminar flow problems by a number of researchers (Pepper, et al, 1990). Pepper et al (1990) introduced a finite element algorithm based on a projection method for calculation of laminar convective fluid flow problems with one point Gauss quadrature rules for all the integrals which allowed calculations of the flow on the slow computers of the PC type. Later, using one of the projection methods, Pepper, et al (1992) calculated laminar flow over a backward-facing step.

The current work is related to comparison studies of the penalty function FEM with the equal-order projection FEM for calculating velocity distributions in a two-dimensional steady incompressible turbulent flow over a backward-facing step. This kind of flows is a widely known benchmark problem for evaluating the ability of a computational fluid dynamics program to accurately predict separated flows. Extensive experimental data for this problem is available from the work of Vogel and Eaton (1984). As can be found in the literature, the standard k-ε model (Launder and Spalding, 1972) with wall functions underpredicts the reattachment length in the backward-facing step problem by the amount in the order of 20-25%. Ignat et al (1998) used an adaptive finite element technique in an attempt to increase the accuracy of the reattachment length predictions using this turbulence model. Their techniques gave improvements to the Stanton number calculations while the usual discrepancy was observed for the reattachment length.

In this study the ability of the penalty function FEM and the equal-order projection FEM to simulate the two-dimensional steady incompressible turbulent flow over a backward-facing step have been investigated and compared. The discrepancies from the calculations were also studied.

## GOVERNING EQUATIONS

The flow regime of interest is modeled by the Reynolds-averaged Navier-Stokes equations in nondimensional form:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla p+ \nabla\cdot\left[\left(\frac{1}{\mathrm{Re}}+\nu_T\right)\left(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T\right)\right]+\boldsymbol{f} \tag{1}$$

$$\nabla\cdot\mathbf{u}=0 \tag{2}$$

The turbulent viscosity $\nu_T$ is computed using the k-ε model of turbulence:

$$\nu_T=C_\mu\frac{k^2}{\varepsilon} \tag{3}$$

The system is closed by including the transport equations for k and ε:

$$\frac{\partial k}{\partial t}+\boldsymbol{u}\cdot\nabla k=\nabla\cdot\left[\left(\frac{1}{\mathrm{Re}}+\frac{\nu_T}{\sigma_k}\right)\nabla k\right] +\nu_T P(\boldsymbol{u})-\varepsilon \tag{4}$$

$$\frac{\partial \varepsilon}{\partial t}+\boldsymbol{u}\cdot\nabla \varepsilon=\nabla\cdot\left[\left(\frac{1}{\mathrm{Re}}+\frac{\nu_T}{\sigma_\varepsilon}\right)\nabla \varepsilon\right] +C_{\varepsilon1}\frac{\varepsilon}{k}\nu_T P(\boldsymbol{u})-C_{\varepsilon2}\frac{\varepsilon^2}{k} \tag{5}$$

where the production of turbulence is defined as

$$P(\mathbf{u})=\nabla\mathbf{u}:\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right) \tag{6}$$

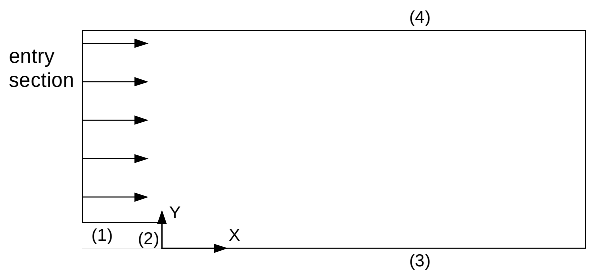
The constants $\sigma_k,\sigma_\varepsilon,C_{\varepsilon1},C_{\varepsilon2},C_\mu$ are set to the standard values proposed by Launder and Spalding (1972).

The Reynolds number is defined as

$$\mathrm{Re}=\frac{|\mathbf{u}|_\infty H}{\nu_\infty} \tag{7}$$

## PROBLEM DESCRIPTION

The problem is a two-dimensional steady incompressible turbulent flow over a backward-facing step. The problem geometry and boundary conditions are shown in Figure1. This problem with the same geometry and boundary conditions was considered by Ignat et al (1998) where mesh adaptation technique was applied. Reynolds number based on the step height is 28,000. At the inlet, Dirichlet boundary conditions are imposed for all variables. The wall functions are used to determine the wall shear stress.




| Entry: | | | Exit: | |
|---|---|---|---|---|
| | $L_e = 4H$ | Wall (1): $L_1 = 3H$ | | $L_s = 5H$ |
| | $u = 1.0$ | Wall (2): $L_2 = H$ | | $u$ = free |
| | $v = 0.0$ | Wall (3): $L_3 = 20H$ | | $v = 0.0$ |
| | $k = 0.02$ | Wall (4): $L_4 = 23H$ | | $k$ = free |
| | $\varepsilon = 0.01524$ | | | $\varepsilon$ = free |

Figure 1. Geometry and boundary conditions for backward facing step.

## WALL BOUNDARY CONDITIONS

The standard k-ε turbulence model is not valid when the turbulent Reynolds number is low. The near-wall region is such an instance. The strategy adopted here uses wall functions which describe the solution near the wall. On solid walls a combination of Dirichlet and Neumann conditions is imposed through wall functions. The imposed wall shear stress is given by

$$\tau_w = u_\tau^2 \tag{8}$$

Furthermore, we take

$$t^* = \tau_w \, sign(\mathbf{u} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}} \tag{9}$$

where $\hat{\mathbf{t}}$ is the boundary tangent unit vector. The friction velocity $u_\tau$ is evaluated from

$$u^+ = \begin{cases} y^+ & y^+ < y_c^+ \\ \dfrac{1}{\kappa}\ln\left(Ey^+\right) & y^+ \ge y_c^+ \end{cases} \tag{10}$$

where $u^+ = u/u_\tau$ and $y^+ = (yu_\tau)/\nu$

Here $u$ is tangential velocity, y the distance to the wall, κ the von Karman constant and E a roughness parameter (E=9.0 for smooth walls). The value of $y_c^+$ is computed to ensure continuity of the velocity profile by solving

$$y_c^+ = \frac{1}{\kappa}\ln\left(Ey_c^+\right) \tag{11}$$

This wall shear stress is supplemented by setting the normal velocity component to zero. The turbulent kinetic energy and its dissipation rate on the mesh boundary are given functions of the friction velocity.

$$k_w = \frac{u_\tau^2}{\sqrt{C_\mu}} \quad ; \quad \varepsilon_w = \frac{u_\tau^3}{\kappa y} \tag{12}$$

## THE PENALTY FUNCTION FINITE ELEMENT METHOD

The continuity equation for incompressible flow is given by $\nabla \cdot u = 0$. The right-hand side of the equation can be modified to introduce pressure such as

$$\nabla \cdot \mathbf{u} = -p/\lambda \tag{13}$$

where λ is a penalty parameter ( $\lambda \rightarrow \infty$ ).

This modified continuity equation provides an explicit relation for pressure:

$$p = -\lambda \nabla \cdot \mathbf{u} \tag{14}$$

The Reynolds-averaged Navier-Stokes equations and the turbulence equations are solved using a finite-element method. Bilinear four-node quadrilateral elements are used to discretize the problem region. The finite element equations are obtained by multiplying the differential equations by suitable test functions and integrating over the domain. Integration by parts is applied for diffusion and pressure gradient terms. Then the Galerkin variational equations for the Navier-Stokes equations will be:

$$\begin{aligned}(\dot{\boldsymbol{u}},\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) &= -(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}) \\ &+ (\boldsymbol{f},\boldsymbol{v}) + \langle t^*,\boldsymbol{v}\rangle\end{aligned} \tag{15}$$

with

$$a(\mathbf{u},\mathbf{v}) = \int_V \left(\frac{1}{\mathrm{Re}} + \nu_T\right)\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right] : \nabla\mathbf{v}\, dV \tag{16}$$

$$\begin{aligned}\langle t^*,\boldsymbol{v}\rangle &= \int_{\partial V\setminus\Gamma_t}\left[\left(\frac{1}{\mathrm{Re}}+\nu_T\right)\left(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T\right)\cdot\boldsymbol{n} - p\boldsymbol{n}\right]\cdot\boldsymbol{v}\;\; ds \\ &+ \int_{\partial V\cap\Gamma_t}\tau_w\cdot\boldsymbol{v}\;\; ds\end{aligned} \tag{17}$$

where $\partial V \setminus \Gamma_t$ denotes either a free stream or outflow boundary and $\partial V \cap \Gamma_t$ represents the portion of the boundary where the law of the wall will be applied to the velocity field.

The expression for pressure above can be substituted into the integral so that pressure will be eliminated.

The Galerkin variational equations will be for the k and ε equations:

$$\int_V \left[ \dot{k}w + \left( \frac{1}{\text{Re}} + \frac{\nu_T}{\sigma_k} \right) \nabla k \cdot \nabla w \right] \text{dV} = \int_V \left[ -\boldsymbol{u} \cdot \nabla k w + \nu_T P(\boldsymbol{u}) w dV - \varepsilon w \right] \text{dV} \tag{18}$$

$$\int_V \left[ \dot{\varepsilon} w + \left( \frac{1}{\text{Re}} + \frac{\nu_T}{\sigma_\varepsilon} \right) \nabla \varepsilon \cdot \nabla w \right] \text{dV} = \int_V \left[ -\boldsymbol{u} \cdot \nabla \varepsilon w + C_{\varepsilon 1} \frac{\varepsilon}{k} \nu_T P(\boldsymbol{u}) w dV - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \ w \right] \text{dV} \tag{19}$$

where the overdot refers to time differentiation. Integrals for all the terms except penalty ones are evaluated numerically using four-point Gauss quadrature rules. For the penalty terms we use reduced integration with one Gauss integration point at the centroid of each element. The solution procedures are listed as follows:

1.given initial condition $u_0, k_0, \varepsilon_0$

2.compute $\nu_T$ from $k_0$ and $\varepsilon_0$

3.for $\nu_T$ given

- (a) solve the Reynolds-averaged Navier-Stokes equations for $u^{n+1}$
- (b) solve the k equation
- (c) solve the ε equation
- (d) solve the modified continuity equation for elementwise constant pressure at the centroid of each element.
- (e) Update $\nu_T$ and go to step 3.

## THE EQUAL-ORDER PROJECTION FINITE ELEMENT METHOD

Here we also use bilinear four-node quadrilateral elements to discretize the problem region. And the Galerkin weighted residual technique is applied for obtaining the solution. Integration by parts is applied for diffusion terms and pressure gradient terms.

The pressure is obtained from the discrete Reynolds-averaged Navier-Stokes equations and a time-differenced version of the continuity equation. The procedures are listed as follows:

1.given initial condition $u_0, k_0, \varepsilon_0$

2.compute $\nu_T$ from $k_0$ and $\varepsilon_0$

3.for $\nu_T$ given

(a) solve the Reynolds-averaged Navier-Stokes equations for $u^*$.

$$\left( \dot{\boldsymbol{u}}^*, \boldsymbol{v} \right) = \left( p^n, \nabla \cdot \boldsymbol{v} \right) - \left( \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n, \boldsymbol{v} \right) - a\left( \boldsymbol{u}^n, \boldsymbol{v} \right) + \left( \boldsymbol{f}^n, \boldsymbol{v} \right) + \left\langle t^{*n}, \boldsymbol{v} \right\rangle \tag{20}$$

(b) solve the potential function equation for Φ based on the $u^*$ values.

$$\left( \nabla \cdot \Phi, \nabla \cdot \boldsymbol{v} \right) = \left( \nabla \cdot \boldsymbol{u}^*, \boldsymbol{v} \right) \tag{21}$$

(c) update the velocity

$$\left( \bar{\boldsymbol{u}}^{n+1}, \boldsymbol{v} \right) = \left( \bar{\boldsymbol{u}}^n, \boldsymbol{v} \right) + \left( \nabla \cdot \Phi, \boldsymbol{v} \right) \tag{22}$$

(d) calculate pressure from a Poisson equation of the form

$$\left( \nabla \cdot p^{n+1}, \nabla \cdot \boldsymbol{v} \right) = \left( \nabla \cdot \tilde{\boldsymbol{u}}, \boldsymbol{v} \right) \tag{23}$$

where

$$\tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}}^{n+1} - \Delta t \{ \nabla p^n - \ \bar{\boldsymbol{u}}^{n+1} \cdot \nabla \bar{\boldsymbol{u}}^{n+1} + \nabla \cdot \left[ \left( \frac{1}{\text{Re}} + \nu_T^n \right) \left( \nabla \bar{\boldsymbol{u}}^{n+1} + \nabla \left( \bar{\boldsymbol{u}}^T \right)^{n+1} \right) \right] + \boldsymbol{f} \} \tag{24}$$

$\Delta t$ - the time step size;

(e) solve the k and ε equations

$$\int_V \dot{k} w \text{dV} = \int_V [ -\boldsymbol{u} \cdot \nabla k w + \nu_T P(\boldsymbol{u}) w dV - \varepsilon w + \left( \frac{1}{\text{Re}} + \frac{\nu_T}{\sigma_k} \right) \nabla k \cdot \nabla w ] \text{dV} \tag{25}$$

$$\int_V \dot{\varepsilon} w \text{dV} = \int_V [ -\boldsymbol{u} \cdot \nabla \varepsilon w + C_{\varepsilon 1} \frac{\varepsilon}{k} \nu_T P(\boldsymbol{u}) w dV - C_{\varepsilon 2} \frac{\varepsilon^2}{k} w + \left( \frac{1}{\text{Re}} + \frac{\nu_T}{\sigma_\varepsilon} \right) \nabla \varepsilon \cdot \nabla w ] \text{dV} \tag{26}$$

(f) update $\nu_T$ and go to step 3.

The integrals are evaluated numerically using four-point quadrature rules.

**RESULTS**

Figures 2-13 show results from the computations. Those are displayed as contour lines for the whole domain and velocity vector distributions behind the step. For both the penalty code and the equal-order code non-uniform mesh of 3015 nodes was applied. The mesh was refined in the regions where high gradients of the solution were expected, that is near the step corner, behind the step and along all the solid wall where the boundary layer must be captured. The Reynolds number based on the step height was 28,000. Dimensions of the channel as well as the boundary conditions for k and ε were taken from Ignat et al (1998) who calculated the problem in an adaptive fashion. Their adaptation technique refined the mesh behind the step to the very high extent though the mesh near the upper wall remained almost unrefined. In this study we refined the mesh on the upper wall manually and therefore the boundary layer in our case was captured by both of the methods well. Below are the contours from the equal-order and finite-element FEM methods for u and v components of the velocity shown in Figures 2-5; TKE shown in Figures 6 and 7; its dissipation rate shown in Figures 8 and 9; and turbulent viscosity shown in Figures 10 and 11. These results are somewhat close. Figures 12 and 13 show the plot of the velocity vector distributions behind the step for both of the methods completes the article. From Figure 13 we see that in the penalty approach the velocity vectors are not distributed parallel to the wall. In our view, it happens because the penalty approach only approximates the incompressibility which is not absolutely valid for turbulent flows.

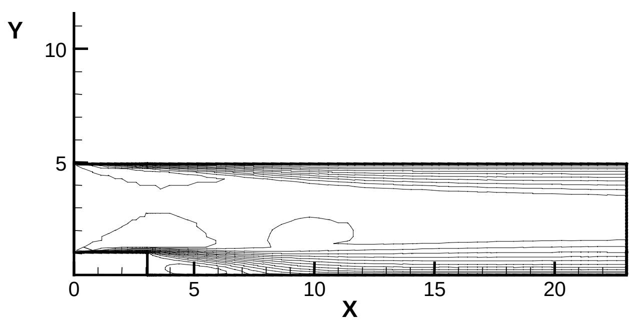



Figure 2. U-velocity contours using equal-order method

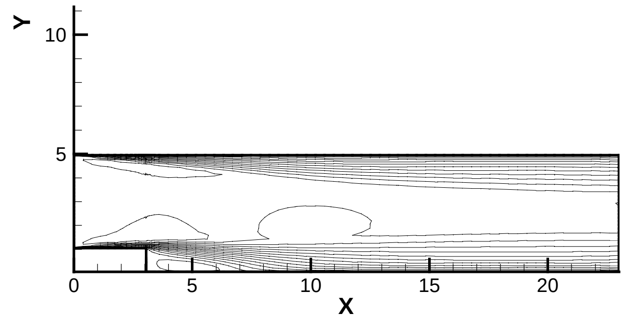



Figure 3. U-velocity contours using penalty method

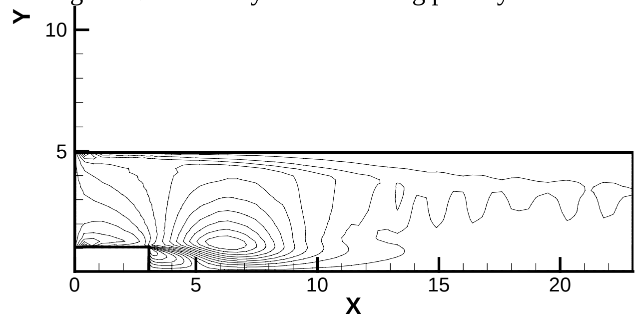



Figure 4. V-velocity contours using equal-order method

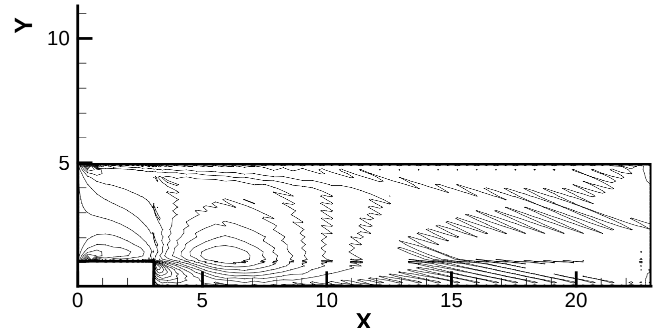



Figure 5. V-velocity contours using penalty method

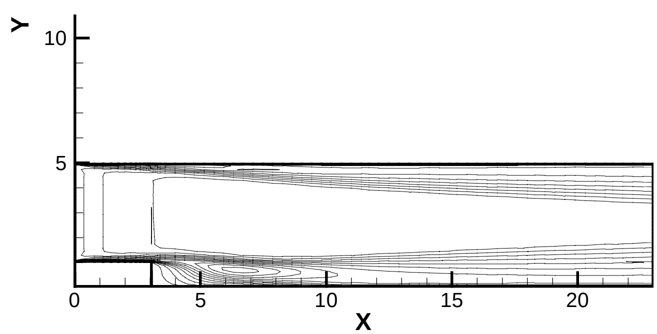



Figure 6. Turbulent kinetic energy (k) contours using equal-order method

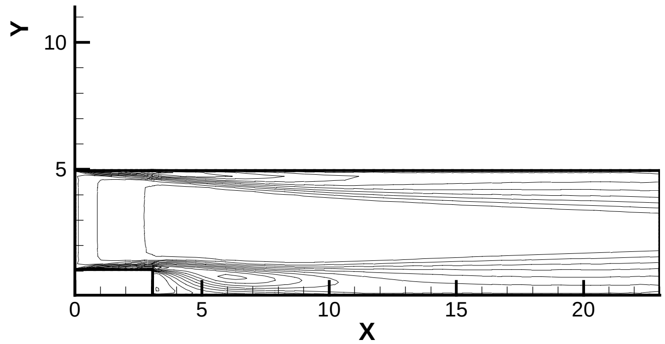



Figure 7. Turbulent kinetic energy (k) contours using penalty method

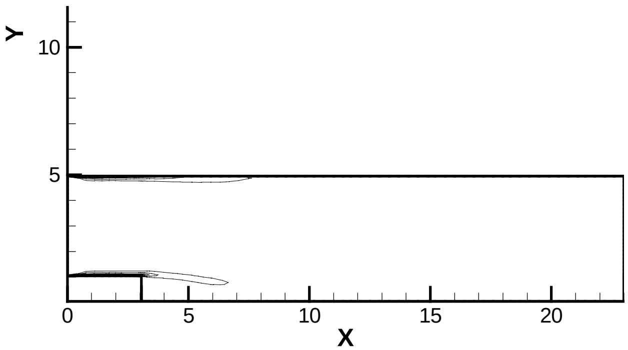



Figure 8. Energy dissipation (ε) contours using equal-order method

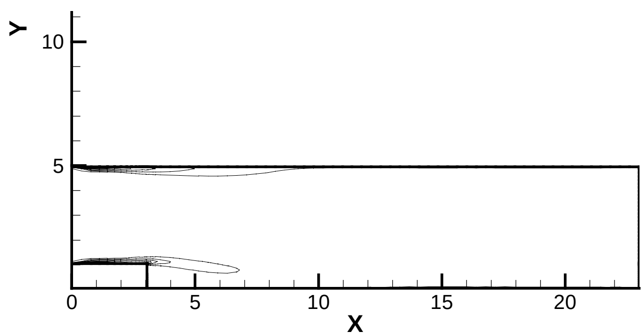



Figure 9. Energy dissipation (ε) contours using penalty method

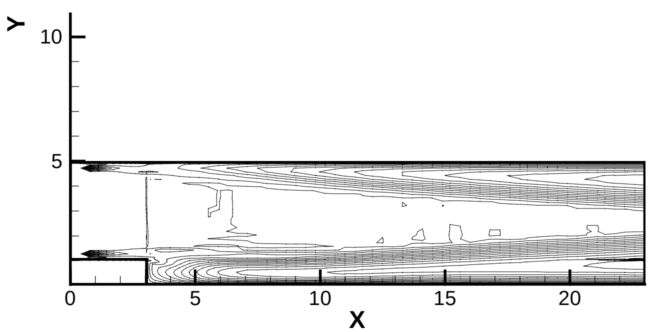



Figure 10. Turbulent viscosity ($\mu_T$) contours using equal-order method

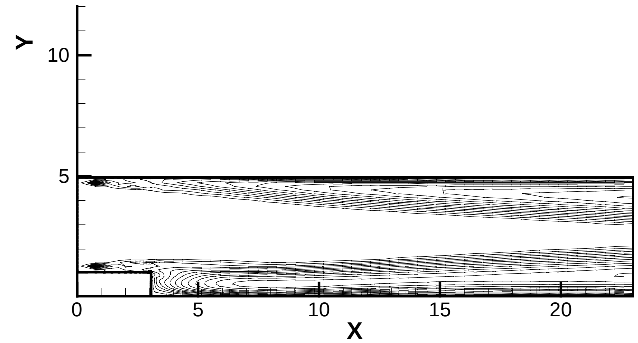



Figure 11. Turbulent viscosity ($\mu_T$) contours using penalty method

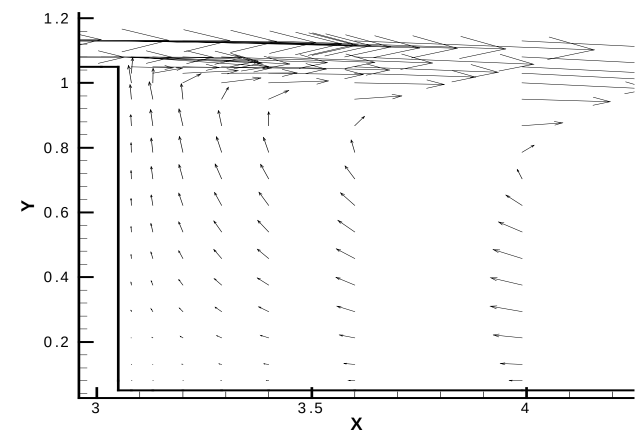



Figure 12. U-velocity circulation contours at the corner after step using equal-order method

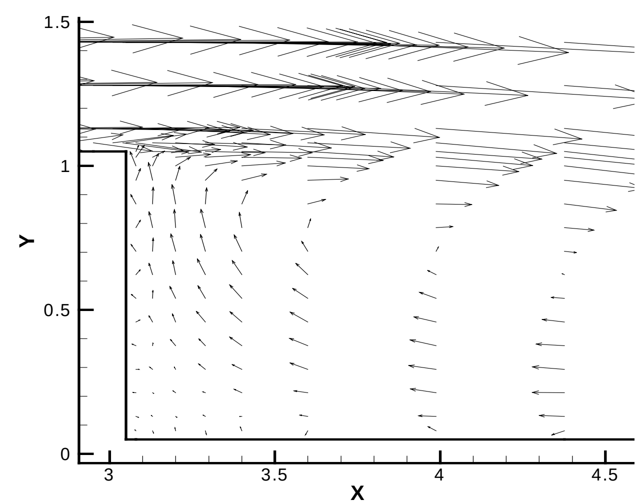



Figure 13. U-velocity circulation contours at the corner after step using penalty method

## REFERENCES

1.Chorin, A.J., 1968, "Numerical Solution of the Navier-Stokes," *Math. Comput.*, **22**, pp.745-762.

2.Gresho, P.M., Chan, S.T., Lee, R.L., Upson C.D., 1984, "A Modified Finite Element Method for Solving the Time-Dependent Navier-Stokes Equations," *Int. J. Num. Meth. Fluids*, **4**, pp.557-598.

3.Hood, P., Taylor C., 1974, "Navier-Stokes Equations using Mixed Interpolation," *Finite Element Methods in Flow Problems*, pp. 121-132.

4.Ignat, L., Pelletier D., Ilinka F., 1998, "Adaptive Computations of Turbulent Forced Convection," *Numerical Heat Transfer*, **34**, Part A, pp.847-871.

5.Patankar, S.V., 1980, *Numerical Heat Transfer and Fluid Flow*, Hemisphere, Washington D.C.

6.Patankar, S.V., Spalding D.B., 1972, "A calculation procedure for Heat, Mass and Momentum Transfer in Three-Dimensional Parabolic Flows," *Int. J. Heat Mass Transfer*, **15**, pp. 1787-1806.

7.Pepper, D.W., Singer, P.S., 1990, "Calculation of Convective Flow on the Personal Computer using a Modified Finite-Element Method," *Numerical Heat Transfer*, **17**, Part A, pp. 379-400.

8.Pepper, D.W., Burton, K.L., Brueckner, F.P., Blackwell, B.F., 1992, "Numerical Simulation of Laminar Flow with Heat Transfer over a Backward Facing Step," *ASME*, **222**, pp.21-26.

9.Vogel , J.J., Eaton, J.K., 1984, "Heat Transfer and Fluid Mechanics Measurements in Turbulent Reattaching Flow Behind a Backward-Facing Step," *Tech. Rep. MD44*, Thrmoscience Division, Mechanical Engineering, Stanford University.

10.Zienkiewicz, O.C., Taylor, R.L., 1989, *The Finite Element Method*, McGraw-Hill, New York.

11.Launder, B.E., Spalding, D.B., 1972, *Mathematical models of turbulence*, Academic Press, London.